ORC 69-23
AUGUST 1969

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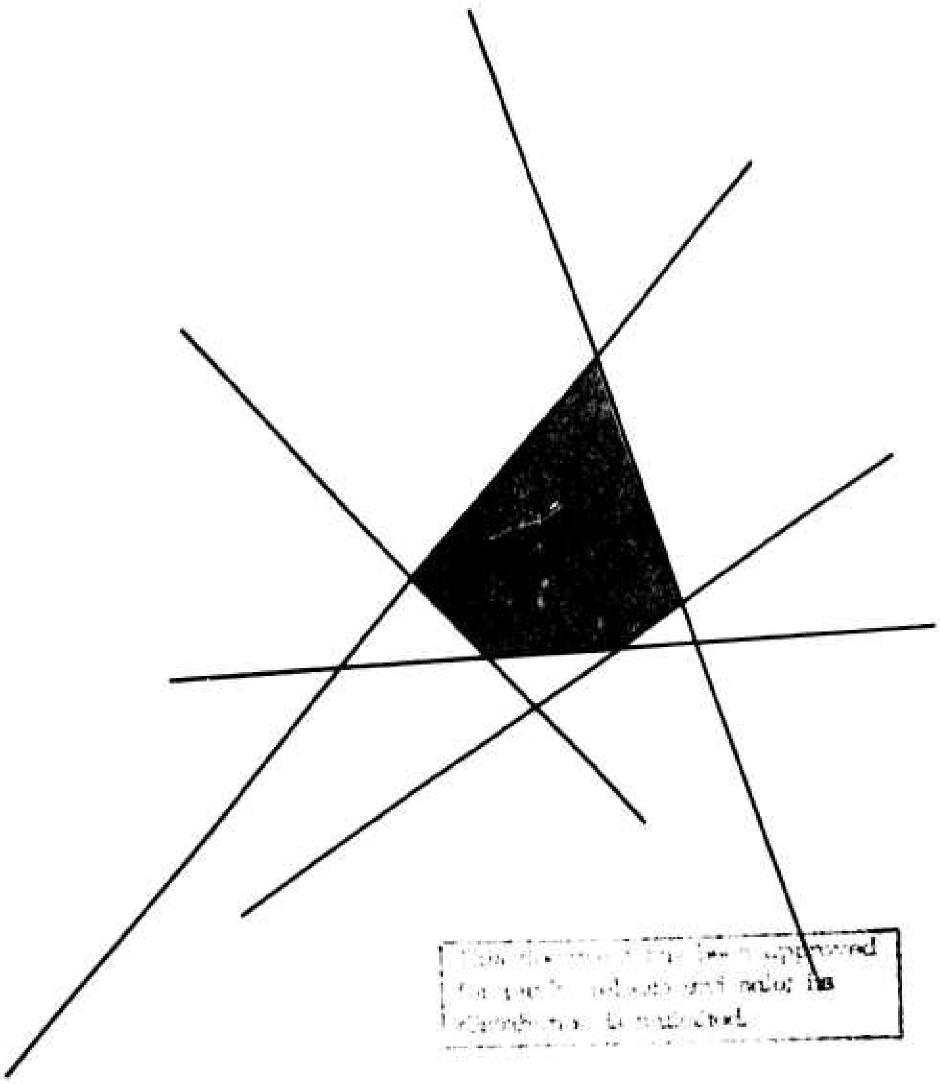

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[+]This work was done while the author held a Senior Postdoctoral Fellowship from the National Science Foundation at the University of Copenhagen, Denmark.

ABSTRACT

The paper analyzes one-sector models of general equilibrium over an infinite time horizon in which there are an infinite number of agents, these being the members of successive generations. It is argued that such models are more realistic than the customary equilibrium models with finitely many agents. Three aspects of these models are then examined including: (1) non-optimalities in the form of productive inefficiency (as noted by Samuelson and others) and consumption non-optimality due to "inflationary" equilibria, (2) the role of "credit" and its effect on the equilibrium is discussed and (3) it is shown that for simple exchange models steady state equilibria are possible with a permanent imbalance of trade in which one country constantly exports to the other, a phenomenon which cannot occur in the classical model.

Price Equilibrium for Infinite Horizon Economic Models

by

David Gale

1. Introduction

This article has a dual purpose. Its immediate aim is to give further results on infinite horizon or "open ended" equilibrium models of the type first studied by Samuelson [7] and later elaborated on by Diamond [6] and Cass and Yaari [3],[4]. Like those works, it will pay special attention to the non-optimalities which can occur in such models and which have been termed "paradoxical" since they contradict well known properties of the classical, i.e. finite horizon model. It also presents some discussion of the role of "credit" in such models and its effect on the optimality properties of equilibrium. This is similar to the treatment by Samuelson and Cass, Yaari of the "contrivance of money" and is probably also related to Diamond's "national debt" in a somewhat more general and abstract framework. In the final section we consider some very simple two country "international trade" models and make the somewhat surprising observation that (for the infinite horizon case) steady state equilibria are possible with a permanent imbalance of trade, where one country constantly exports to the other.

The article also has a propaganda purpose. I do not claim that the non-optimalities and trade imbalances exhibited here correspond to or even necessarily throw much light on the non-optimalities and imbalances which actually plague real economies. It does seem possible, though, that further investigations of more realistic open ended models may help our understanding of such real phenomena. It strikes me as significant, for example, that even in these very simple models one is led at once to consider notions like volume of savings, stock of capital, level of credit, - concepts which are familiar in everyday economic experience. For these reasons I hope the results presented here, fragmentary though they are, may stimulate others to get into the infinite horizon game.

Of course, as mentioned, the popular theorems of static equilibrium theory concerning (Pareto) optimality and the core fall by the wayside. This, in my opinion, is all to the good. It is, after all, non-optimality rather than optimality in economic life which people are really concerned about and it may be a hopeful thought that it is possible to have a theory of non-optimality within the framework of a general equilibrium theory. Further, as a reflection of reality the finite horizon model is almost certainly empirically wrong. A society does not in fact make its economic decisions based on some finite time horizon, (individual "economic agents" probably do make decisions based on finite horizons but the horizon then varies with the agent, which is the whole point in the models to follow). Finally, I suspect that the usual optimality theorems are not even operationally meaningful in the context of a finite horizon theory. What would one mean by saying, for example, that the present state of an economy is optimal? Presumably one would mean that society could not be achieving greater satisfaction without going to a less desirable program of capital accumulation, and conversely a better accumulation program could not be achieved without sacrificing present satisfaction. But what is meant by a better or worse accumulation program? Obviously a program is better or worse according as it can provide more or less satisfaction to society in the future, but if one accepts this one is again forced to consider an infinite not a finite horizon theory.

I want to say one more word about non-optimalities. There are many economists who feel that the problems of real economics can not be analyzed by means of any equilibrium theory. Professor Frank Hahn has said quite explicitly that he believes the understanding of these problems may have to await the development of a suitable non-equilibrium economic theory. I am certainly not prepared to take issue with this point of view but if it is correct it is bad news indeed for economic theorists, for no one, as far as I am aware, has the slightest clue as to what a satisfactory non-equilibrium theory might look like. The equilibrium concept has certainly been the cornerstone of economic theory to date and for purely practical reasons it would seem worthwhile to continue to give it every chance before it is abandoned in favor of unexplored territory. This is one more reason for trying to find out what can be done with the more flexible open ended models we are about to describe.

2. The Classical Model

Since part of our purpose is to contrast infinite horizon models with the traditional ones we shall here give a very brief formulation of a rather general homogeneous (constant returns to scale) model and derive the usual (and trivial) results on optimality of equilibrium. In the later sections we will see precisely at what point and for what reasons these theorems break down in the infinite horizon case.

We consider an economy in which there are n goods and m agents. Associated with the i^{th} agent is a subset V_i of n-space called the agent's opportunity set. A vector v_i from V_i represents amounts of various goods which the agent is able to supply or demand. For the economy as a whole there is a subset T on n-space called the technology which has the homogeneity property

(2.1) For $v \varepsilon T$ and $\lambda \geqq 0$ $\lambda v \varepsilon T$.

In current terminology, the set T consists of all the activities which are possible for the model. These may be thought of as including not only the usual production and consumption activities but also such things as transportation, education, entertainment, various forms of recreation, in fact, all the things with which people occupy their time.

A state (v_i) of the model consists of m vectors $(v_1, \ldots, v_m)$ where $v_i \varepsilon V_i$ and

(2.2) $\Sigma v_i \varepsilon T$. (Σ means $\sum_{i=1}^{m}$)

This says that the supplies and demands of individual agents must be capable of being satisfied by a technologically feasible activity.

To complete the description of the model we assume that the i^{th} agent has a preference ordering denoted by $\succ_i$ defined on vectors in his opportunity set V_i. (We assume the interpretation of these concepts is familiar.) A state $(\bar{v}_i)$ is called (Pareto) <u>optimal</u> if there is no other state (v_i) such that $v_i \succ_i \bar{v}_i$ for all i.

<u>Definition</u>: A <u>price equilibrium</u> consists of a state $(\bar{v}_i)$, an n-vector (prices) $p = (\pi_1,\ldots,\pi_n)$ and an m-vector (income transfers) $d = (\delta,\ldots,\delta_m)$ satisfying:

(I) (consumption condition) $\bar{v}_i$ maximizes $\succ_i$ subject to

$$p \cdot v_i = \delta_i$$

(II) (production condition) Σv_i maximizes $p \cdot v$ among all v in T.

These are the usual conditions asserting that consumers maximize satisfaction subject to their budget equation and producers select the most profitable activity available. Condition (II) can be given the equivalent form

(II') $p \cdot v \leqq 0$ for all v in T and $p \cdot \Sigma \bar{v}_i = 0$

The inequality of (II') follows from homogeneity since if $p \cdot v > 0$ for some v then $p \cdot \lambda v$ would have no maximum as a function of λ. The last part of (II') follows because T contains the zero vector.

A price equilibrium is called laissez-faire in the special case when $d = 0$.

We now derive three simple properties of an equilibrium

Property 1. In any price equilibrium $\sum_{i=1}^{m} \delta_i = 0$

Proof. From (I) and (II') $\sum_{i=1}^{m} \delta_i = \sum_{i=1}^{m} p_i \bar{v}_i = p \cdot \Sigma \bar{v}_i = 0$.

Property 2. An equilibrium state $(\bar{v}_i)$ is optimal.

Proof. If (v_i) is a state with $v_i \succ_i \bar{v}_i$ then from (I) $p \cdot v_i > \delta_i$ so $\Sigma p \cdot v_i > \Sigma \delta_i = 0$ from Property 1, but this contradicts (II').

To describe the third property we must generalize the notion of state. Let S be a subset of the indices $1, 2, \ldots, m$. An S-state is a set of vectors v_i, $i \in S$ and $\sum_S v_i \in T$.

Definition: A state $(\bar{v}_i)$ is in the core if there is no subset S and an S-state (v_i) such that $v_i \succ_i \bar{v}_i$ for all i in S.

Property 3. A laissez-faire equilibrium state is in the core.

Proof. If $v_i \succ_i \bar{v}_i$ for i in S then $p \cdot v_i > 0$ for i in S and $p \cdot \sum_S v_i > 0$ again contradicting (II').

Properties 2 and 3 have been central in much of the literature on equilibrium. They are, of course, trivial consequences of the definition and the interest has been not in them but in various converse statements. By way of a historical footnote we recall that a converse of Property 2 was given by Arrow [1] in which it is shown that under suitable connexity assumptions on the sets V_i, T and the preferences $\succ_i$ every optimal state is the state of some price equilibrium. Converses of Property 3 have been given by Scarf [8], Debreu and Scarf [5], Vind [9] and others and are to the effect that if the number of agents is large in an appropriate sense then the states of the core are "close to" the equilibrium states.

Property 1 has heretofore been considered so obvious as not to require special mention. It justifies the term transfers in describing the numbers δ_i. We emphasize it here, however, because like the other two properties it fails to hold in the open ended case, and this will play an important role in the later analysis.

3. The Samuelson Model

We turn now to infinite horizon models which can be divided into two types according to whether or not the agents are mortal or immortal. The second approach has been used for example by Arrow and Kurz [2]. It has, however, some disadvantages (aside from the fact that it runs counter to our knowledge of biology). It requires preference orderings on spaces of infinite sequences and, more seriously, there are considerable difficulties in defining an appropriate budget constraint. Both these difficulties vanish for the case of mortal agents of which, however, we now need an infinite number. The simplest non-trivial model of this sort appears to be the one first formulated by Samuelson which we now recall. The properties to be described will be common to all the models discussed hereafter.

The economy involves a single good which can either be consumed yielding utility or invested yielding more of itself. People (agents) in the model live for two periods and receive an amount of goods w (w for (real) wage) only during the first period. We may think of them as actually creating the goods from their labor while they are young and vigorous. Let us call people born in period t the members of generation t and denote them by G_t. A member of G_t will consume amounts c_t and c_t' in periods t and $t+1$ so his opportunity set will be the positive quadrant in the plane. If p_t and p_{t+1} are the prices of goods in these same periods then consumers will want to maximize satisfaction subject to the budget equation

(3.1) $$p_t c_t + p_{t+1} c_t' = p_t w_t$$

It will usually be more convenient to divide (3.1) by p_t and obtain the equivalent form

(3.1)' $$c_t + c_t'/\rho_t = w_t$$

Here $\rho_t = p_t/p_{t+1}$ and is the usual interest factor. We will also very definitely want to include the possibility of income transfers in which case (3.1)' takes the more general form

(3.2) $$c_t + c_t'/\rho_t = w_t + \delta_t$$

As in the previous section (3.1)' will be called the laissez-faire case.

If we wish to solve the consumer's problem for a particular model we must have an explicit description of his preferences which is conveniently done using the usual utility function. An interesting special case is given by the function

$$(3.3) \qquad u(c_t, c_t') = (1-\sigma)\log c_t + \sigma \log c_t' \qquad 0 \leq \sigma \leq 1.$$

Maximizing (3.3) subject to (3.2) gives

$$(3.4) \qquad c_t = (1-\sigma)(w_t + \delta_t), \qquad c_t' = \rho_t \sigma_t (w_t + \delta_t)$$

(we leave to the reader the job of doing the required calculus). This is then the case where consumers save the fraction σ of their income independent of prices, income being the sum of the wage and transfer payment.

4. Productively Inefficient Equilibria

Nothing was said in the previous section about production. Let us now make the very simple assumption that if a unit of goods is stored (invested) it deteriorates leaving λ units one period later where $0 < \lambda < 1$. (Our example is a very slight variation of that of Cass and Yaari [4] who assumed an increasing population rather than deteriorating goods. In fact part of this section will be covering the same ground as [4] but with a somewhat different emphasis looking ahead to the next section.) We suppose that all people of all generations have identical tastes. We choose units so the wage w is unity and consider first the case where no trade is permitted. An individual will then consume an amount c during his youth and save the amount $s = 1-c$ which deteriorates to $\lambda(1-c)$ and is consumed as c' in the next period. He will therefore choose (c,c') to maximize his satisfaction subject to the equation

$$(4.1) \qquad c + c'/\lambda = 1.$$

We allow any preference ordering assuming only that it gives $c' > 0$ (thus we eliminate the unlikely situation in which people are prepared to starve in their old age). The point is then that any such program without trade will be productively inefficient (and a fortiori non-optimal), for in each period people will store the positive amount $s = (1-c)$ and obtain an output λs so the net amount of goods available for consumption in each period is $1 - (1-\lambda)s$. But this means that the economy is "throwing away" the amount of goods $(1-\lambda)s$ in every period, since the endowment of the economy is one unit (per young person) in each generation.

The point of the model is that introducing prices does not help the situation in the laissez-faire case and thus, competitive equilibrium is also productively inefficient. Referring to budget equation (3.1)' we claim that at equilibrium $\rho_t = \lambda$ for all t. Namely we cannot have $\rho_t < \lambda$ for this would be the case of positive profits. People could achieve any consumption program (c,c') as follows

(i) buy $(\lambda c+c')/(\lambda-\rho_t)$ in the first period

(ii) consume c and store the remaining $(c'+\rho_t c)/(\lambda-\rho_t)$ giving

(iii) $\lambda(c'+\rho_t c)/\lambda-\rho_t$ next period. Consume c' leaving

(iv) $\rho_t(c'+\lambda c)/(\lambda-\rho_t)$,

which will exactly pay for the amount (i). Such a program is of course not feasible giving an excess demand. On the other hand if $\rho_t > \lambda$ then people after consuming the amount c will wish to sell $(1-c)$ in their youth so as to consume $\rho_t(1-c)$ in their old age (rather than $\lambda(1-c)$ they would get by storing). But since everyone will want to sell there will be an excess supply and again no equilibrium. Thus, equilibrium requires $\rho_t = \lambda$ so (3.1)' becomes (4.1) and we are back to the inefficient program.

It is easy to see where the proof of the properties of the classical model breaks down. Those proofs all required summing the budget equation of all the agents in the model, but in the present case the number of agents is infinite so one gets a meaningless equality of two divergent infinite series. In this connection it is worth looking at what happens if instead of considering an infinite horizon we consider a very long finite horizon, say, a thousand generations. Then, of course, the equilibrium program and indeed the program without trade is efficient and Pareto optimal, but it is a very "unfair" optimum for it requires the first 999 generations to make a substantial sacrifice in consumption solely for the benefit of the final generation. It seems to me that the non-optimality which the infinite horizon approach displays gives the more accurate description of the situation.

The "cure" for these non-optimalities has been discussed by Samuelson in terms of "the contrivance of money" and by Cass and Yaari under "financial intermediation" between generations. Diamond for a somewhat different model (see Section 6) considers the effects of national and internal debt. At the present level of abstraction we will not consider any specific type of economic institution but will look only at the effect of such measures which is to change the right hand side of the consumers' budget equation. I am grateful to Prof. Vind for pointing out to me the fact that this is the essence of the situation.

Let us return to the model just considered and for convenience look at the special case of (3.3), (3.4) in which consumers always save the fraction σ of their income. We claim that if income transfers are allowed then the following sequence of transfers and interest factors will yield an equilibrium which is also optimal. In the first period $\rho_1 = 1-\sigma$ and $\delta_1 = \sigma/(1-\sigma)$. Thereafter $\rho = 1$ and $\delta = 0$. The total income to G_1 will then be $1/1-\sigma$ of which it consumes 1 unit in the first period, saves the remaining $\sigma/(1-\sigma)$ which at interest factor ρ_1 is worth σ in the next period. Thereafter every generation consumes $(1-\sigma,\sigma)$. Since total consumption of old and young together in each period is one the program is efficient and optimal. (This is just one of infinitely many ways in which optimality can be achieved by proper pricing. It would be interesting to see whether Arrow's Theorem holds for this model so that any optimal program can be induced by suitable prices and transfers.)

The main conclusions of this section are then,

(A) For infinite horizon models a price equilibrium need not be optimal. In the present example the laissez-faire equilibrium is non-optimal. Thus Properties 2 (and hence 3) of Section 2 fail to hold.

(B) By a suitable system of income transfers the optimality of equilibrium can be restored. However, such a system need not satisfy Property 1 of Section 2. In the example just considered, for instance, we have $\sum_{t=1}^{\infty} \delta_t = 1$.

<u>Remarks</u>: Because of (B) above the term "transfers" no longer seems appropriate to describe the δ_t. The amount δ_1 added to G_1's income is not taken away from anyone else. Further, the amount δ_1 though measured in real (goods) units cannot be handed out in the form of goods, since there is only one unit of goods in the economy in period one and this is entirely used up in the wage. The amount δ_1 is therefore a supplement in the form of a <u>credit</u> to G, which can be used for future consumption. This "fictitious" addition to G_1's income works because of the fact that G_1 wants to save anyhow. Should G_1

suddenly change its collective mind and demand immediate payment in goods the demands could not be met. This is reminiscent of the familiar fact that banks are not required to carry sufficient funds to cover the accounts of all their depositors.

It is now clear how money might enter the picture. The first generation might be handed the amount of money $\delta_1 = \sigma/(1-c)$ (by a "monetary authority") which because of the negative interest rate becomes worth only σ in the next and subsequent periods. This amount is then handed down from one generation to the next ad infinitum. In this interpretation one might think of the numbers δ_t as being under the control of some central authority, and the problem of public finance becomes that of suitably regulating the right hand side of people's budget equation, which is perhaps not an unreasonable way to look at it. However, there is nothing sacred about this particular interpretation. An equally attractive "story" for our example might be that the "workers" of G_1 were able to negotiate for themselves a wage increase δ, which resulted in the price increase ρ_1 in the next period after which things settled into the steady state.

One further point should be made. Recall in the laissez-faire case it was argued that we cannot have $\rho_t > \lambda$, for then everyone would want to sell, giving an excess supply as there are no buyers. If one creates buyers artificially by "subsidizing" the old people or by having the "government" buy up whatever the young people don't wish to consume, then from then on the laissez-faire equilibrium proceeds smoothly with transfers and interest rates equal to zero. Once this has been managed by one means or another the market mechanism takes care of things for ever after.

5. Inflationary and Deflationary Equilibrium

In this section we shall exhibit another type of non-optimality which was suggested to me by Karl Vind. We consider the model of the previous section and look for equilibria which are (a) productively efficient and (B) stationary, meaning that prices, transfers and consumptions are all constant. Because of (A) this means that the stationary consumption (c,c') must satisfy

(5.1) $\quad c + c' = 1$.

FIGURE 1

The situation can best be analyzed by referring to Figure 1 in which the graph of equation (5.1) is given by the line AB. The point 0 clearly corresponds to the optimal steady state consumption and can be achieved by an equilibrium with $\rho = 1$, $\delta = 0$, as noted in the previous section (provided one has a mechanism for getting it started). Now, however, since we permit non-zero transfers there are other possible equilibria. In fact if we make the usual assumption about convexity of indifference curves, then any point 0' on AB below the point 0 corresponds to any equilibrium. To see this consider the tangent line A'B' to the indifference curve at 0' and let the equation of this curve be

$$(5.2) \qquad c + c'/\rho = 1 + \delta$$

The number ρ is determined by the slope of the indifference curve at 0' and is the "coefficient of substitution" between present and future consumption, i.e. the rate at which one must replace future by present consumption in order to maintain the same level of satisfaction. From the figure this number is seen to be greater than one. It follows therefore from (5.1) that δ must be negative.

What is the economic picture that emerges from this? The negative δ could be thought of as a government tax or in the collective bargaining interpretation it might mean that employers are able to squeeze the workers lowering wages by the amount δ. In any case the effect is deflationary for ρ is greater than one so prices are falling with the result that people are badly off in their youth and better off in their old age. The deflationary equilibrium is "bad" because people would prefer in the steady state to be at 0 rather than 0'. On the other hand, this deflationary equilibrium is optimal once it has been started for there is no way to increase the consumption of the young people without taking something away from the old. To get out of the deflationary trap requires a sacrifice by at least one generation of old people.

The point 0" represents the opposite situation. Here δ is positive (the unions have forced up the wages, say) so ρ is less than one (prices are increasing) and the older people are being squeezed since their savings have less value because of the price inflation. Note that not every point above 0 on AB gives an equilibrium since ρ cannot be allowed to drop below λ, the deterioration factor for then, one would have the unstable situation of positive profits. Now the inflationary equilibrium, unlike the deflationary case, is not Pareto optimal. It is possible to move from 0" to 0 without exacting a sacrifice from anyone. It is only necessary for the young people of one generation to hand some goods over to their parents. For this act of generosity they can be more than compensated, utility-wise, in their own old age. From the equilibrium point of view one easily sees what to do. One must decrease the right hand side of the budget equation. An appropriate tax, for instance, will do the job. Tax the young people of one generation and subsidize their parents. From then on make sure that total income to the young stays at one.

This example is of course extremely naive, yet it does point up in a nice way the importance of the numbers δ which act as the "control variables" in determining the dynamic behavior of the model. We note finally that Property 1 of static models breaks down with a vengeance in this example. The quantities δ_t instead of summing to zero sum to plus or minus infinity!

6. Equilibria with Imbalance of Trade

In this section it is shown that for open ended exchange models price equilibrium does not, as is generally believed, imply balance of trade. The models are as always, of course, highly stylized. Nonetheless, they strongly suggest that even for realistic models one should not expect trade balance as a consequence of equilibrium.

Consider first two contries C_1 and C_2 each with a population and technology like those of the preceding sections, the only difference being that the rates of deterioration λ_1 and λ_2 are not the same, say $\lambda_1 > \lambda_2$. For a laissez-faire equilibrium it then follows that the interest factor ρ must equal λ_1. Namely, it cannot be smaller for this would permit positive profits in C_1, and it cannot be larger for then as in Section 3. People in both countries would want to sell whatever they did not consume, producing excess supply.

Let w_2 be the wage in C_2. Then young people in C_2 will consume some amount c_2. The rest (savings) $s_2 = w_2 - c_2$ they will sell to C_1 and in the next period they will buy back and consume the amount $c_2' = \lambda s_2$ (here c_2 and s_2 are, of course, chosen to maximize utility). C_1 neither loses nor gains in this transaction and so it is compatible with equilibrium. But now observe that this pattern of trade is repeated in every period. The young people will be exporting s_2 while the old will be importing s_2 and thus in each period after the first there will be a net export of $(1-\lambda)s_2$ units from C_2 to C_1 (in the first period C_2 exports even more, namely s_2 units). Surely the situation is rather paradoxical. In every period C_2 sends some of its national product out of the country and gets nothing in return. To an observer from outer space this would surely appear absurd. Why should people in C_2 be making this perpetual gift of goods to C_1 when they could be consuming and enjoying it themselves? Of course the people in C_1 are not benefitting either since the amount they receive is simply being stored and allowed to deteriorate. This equilibrium is surely not in the core (Property 3), since C_2 alone could be doing better by not exporting at all. On the other hand, observe that by engaging in trade C_2 is better off than it would be under its own laissez-faire equilibrium since the goods it saves are deteriorating at the rate λ_1 rather than λ_2.

One might complain that the examples discussed so far have had rather non-typical production positions in that goods deteriorate rather than grow. This could have been avoided by considering durable goods and an expanding population. In any case, our final example will show that trade imbalance can exist even for "productive" technologies, of the type considered by Diamond [6].

Consider first a single country in which population behaves as in the earlier examples. For convenience we assume people's utility function has the form (3.3) in that they always save the fraction σ of their income. Goods in this case are not produced from labor alone but from labor and goods together through a neo-classical production function f, where $f(x)$ is the output from x units of input at full employment of (one unit of) labor. We assume that goods are durable so that if x units are invested at time t then $f(x) + x$ becomes available at $t+1$. For such a model prices are determined by the technology for we assume that output is homogeneous as a function of input and labor together. The production condition (II') of Section 2 then requires the profits be zero at the equilibrium value of input and non-positive for all other values. Letting p_t, p_{t+1} and W_t be prices in successive periods and money wages we have

$$p_{t+1}(f(x_t)+x_t) - p_t x_t - W_t = 0$$

and dividing through by p_{t+1} gives

(6.1) $$f(x_t) - r_t x_t - w_t = 0$$

where $r_t = p_t/p_{t+1} - 1$ which is the interest rate, and w_t is the real wage. Also x_t must maximize (6.1) so differentiating gives $r_t = f'(x_t)$ and hence

(6.2) $$w_t = f(x_t) - x_t f'(x_t).$$

Finally young people according to our assumption consume the fraction $1-\sigma$ of their wage and the rest is saved as input to production in the next period. This gives the simple recursion

(6.3) $$x_{t+1} = \sigma(f(x_t) - x_t f'(x_t))$$

We are interested here only in non-zero stationary solutions of (6.3), that is, positive values of x satisfying

(6.4) $$x = \sigma(f(x) - xf'(x)).$$

In many cases such stationary solutions will exist. For example if $f(x) = Ax^{1-\alpha}$ with $0 < \alpha < 1$ then the stationary solution is $x = (\sigma\alpha A)^{1/\alpha}$.

Now consider two countries C_1 and C_2 of this type with identical production functions, the only difference being that the savings ratios σ_1 are different. Goods are freely transportable between countries so prices and therefore the interest rate is the same in both countries but the interest rate determines the input $(r = f'(x))$ so input x must be the same in both countries and input determines the wages w from (6.2) so these are also the same. Let (c_i, c_i') be the stationary consumption schedule and s_i the amount saved by young people in C_i. Then we have in each country

$$c_i + s_i = w, \qquad c_i' = (1+r)s_i$$

so that total consumption in C_i is given by

$$(6.5) \qquad c_i + c_i' = w + rs_i = w(1+r\sigma_i)$$

and since we are considering a stationary solution we see that total consumption in every period is higher in the country with the higher savings ratio. But inputs and outputs of production are the same in both countries, so the only way for one country to achieve more consumption is for it to import from the other country in every period, again to the bewilderment of the observer from Mars.

The paradox in this case is more apparent than real. Suppose $\sigma_1 > \sigma_2$ and let us refer to the people of C_1 and C_2 as patient and impatient respectively. The young people in C_2 in order to satisfy their impatience are willing to borrow goods from the patient people and pay back the loan with interest in the next period. There is really nothing surprising about this but it looks strange to the outsider especially if he has not seen the first period in which the goods were flowing the other way. In any case this is another phenomenon which can occur in the open ended model but not in the classical case.

We have said nothing in these exchange models about the possibility of introducing income transfers into the various countries budget equations. If one tries to do this one immediately runs into serious and important problems. Suppose one country adds a δ of credit to its consumers budget equation. How can the second country be forced to "recognize" this credit? If the credit is implemented in the form of money we immediately are confronted with problems of exchange rates between currencies. This suggests a vast new range of problems. It is not even clear what the proper notion of equilibrium is for countries with different currencies. In any case it is again a virtue of the open ended model that it brings us face to face with the currency problem and indicates that such problems may have nothing to do with the quantities of precious metals in the different countries but rather they are caused by the difficulties of exchange between countries which domestically may have widely different credit policies.

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Unclassified

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* University of California, Berkeley	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

PRICE EQUILIBRIUM FOR INFINITE HORIZON ECONOMIC MODELS

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Research Report

5. AUTHOR(S) *(First name, middle initial, last name)*

David Gale

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
August 1969	24	9

8a. CONTRACT OR GRANT NO. Nonr-222(83) b. PROJECT NO. NR 047 033 c. Research Project No.: RR 003 07 01 d.	9a. ORIGINATOR'S REPORT NUMBER(S) ORC 69-23 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*

10. DISTRIBUTION STATEMENT

This document has been approved for public release and sale; its distribution is unlimited.

11. SUPPLEMENTARY NOTES NONE	12. SPONSORING MILITARY ACTIVITY Mathematical Sciences Division, Office of Naval Research Washington, D.C. 20360

13. ABSTRACT

SEE ABSTRACT.

DD FORM 1 NOV 65 1473 (PAGE 1)

S/N 0101-807-6811

Unclassified

Security Classification

A-31409

Unclassified

Security Classification

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Competitive Equilibrium						
Pareto Optimality						
Balance of Trade						
Inflation and Deflation						

DD FORM 1 NOV 65 1473 (BACK)

S/N 0101-807-6821

Unclassified

Security Classification